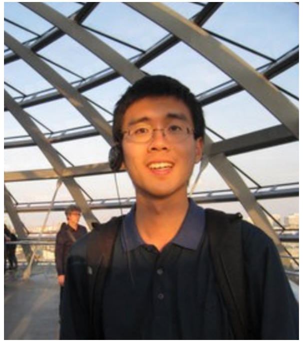
Envy-free house allocation with minimum subsidy

Davin **C**hoo, Yan Hao **L**ing, Warut **S**uksompong, Nicholas **T**eh, Jian **Z**hang





* Alphabetical ordering; Class project of CS6235 AY22/23 Sem 2; Appeared in Operations Research Letters (ORL), 2024

Fair division of indivisible goods

Goods

Agents			
	10	3	7
	10	6	4

Agent utility

u_i(item)

u  () = 3

u  () = 6

https://thenounproject.com/icon/girl-1257314/
https://thenounproject.com/icon/boy-1257319/

https://thenounproject.com/icon/toy-car-809593/
https://thenounproject.com/icon/basketball-2243694/
https://thenounproject.com/icon/gamepad-2243701/

Fair division of indivisible goods

Goods

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	10	3	7
	10	6	4

Envy-free allocation [Fol67, Var74] if we assume *additive* utilities

u ( ) = 6 + 4 = 10 = u ( )

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u_i(item)

u ( ) = 3

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[Fol67] Duncan K. Foley. *Resource allocation and the public sector*. Yale Economics Essays, 1967.
[Var74] Hal R. Varian. *Equity, envy, and efficiency*. Journal of Economic Theory, 1974.

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Agents

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Envy-free allocation [Fol67, Var74] if we assume *additive* utilities

u ( ) = 6 + 4 = 10 = u ( )

	10	2	3
	10	3	2

Envy-free allocation may not exist in general (whoever does NOT get ball will be envious)

u ( ) < u ( )

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[Var74] Hal R. Varian. *Equity, envy, and efficiency*. Journal of Economic Theory, 1974.

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Envy-free allocation [Fol67, Var74] if we assume *additive* utilities

u ( ) = 6 + 4 = 10 = u ( )

	10	2	3
	10	3	2

There is an envy-free allocation if we allow **incomplete allocation**

u ( ) ≥ u ( )

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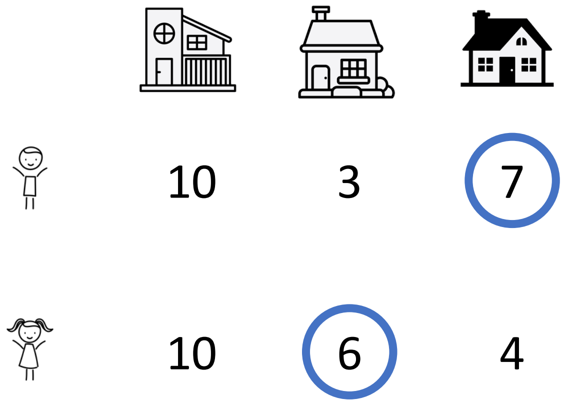
House allocation problem [HZ79, Zho90, AS03]

- m houses
- n agents
- $m \geq n$
- Each agent gets **exactly one** house
  - Complete allocation when $m = n$
  - Incomplete allocation when $m > n$



An incomplete envy-free allocation since $m = 3 > 2 = n$

https://thenounproject.com/icon/house-6659944/
https://thenounproject.com/icon/house-4309135/
https://thenounproject.com/icon/house-6678609/

[HZ79] Aanund Hylland,Richard Zeckhauser. *The efficient allocation of individuals to positions*. Journal of Political Economy, 1979.
[Zho90] Lin Zhou. *On a conjecture by Gale about one-sided matching problems*. Journal of Economic Theory, 1990.
[AS03] Atila Abdulkadiroğlu, Tayfun Sönmez. *Random serial dictatorship and the core from random endowments in house allocation problems*. Econometrica, 2003.

Envy-free relaxations for indivisible goods

- Problem: Envy-free allocation may not always exist
- Common relaxations of envy-free (EF)
  - EF1 [Bud11]: Envy-free up to at most 1 item
    - No longer envy if drop *some* good from other agent's bundle
  - EFX [CKMPSW19]: Envy-free up to at most any item
    - No longer envy if drop *any* good from other agent's bundle

Doesn't make sense in the house allocation problem!

[Bud11] Eric Budish. *The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes*. Journal of Political Economy, 2011
[CKMPSW19] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel Procaccia, Nisarg Shah, Junxing Wang. *The unreasonable fairness of maximum Nash welfare*. ACM Transactions on Economics and Computation (TEAC), 2019

Envy-free relaxations for indivisible goods

- Problem: Envy-free allocation may not always exist
- Common relaxations of envy-free (EF)
  - External subsidy [HS19] ← Envy-free = Zero subsidies required!
    - Total utility = Allocated good utility + given subsidy

	10	3	7
	10	6	4

If we give a subsidy of $3 to and $0 to :

u ( ) + 3 = 10 = u ( ) + 0

u ( ) + 0 = 10 ≥ 7 = u ( ) + 3

[HS19] Daniel Halpern, Nisarg Shah. *Fair division with subsidy*. Symposium on Algorithmic Game Theory (SAGT), 2019

Envy-free allocation with subsidies

- Allocation **a** = $(a_1, \ldots, a_n)$, where each a_i is a distinct house
- Subsidy vector **s** = $(s_1, \ldots, s_n)$, where (finite) $s_i \geq 0$ for all $i \in [n]$
- Outcome (**a**, **s**) is envy-free if

$$u_i(a_i) + s_i \geq u_i(a_j) + s_j \qquad \text{, for every pair of agents } i, j \in [n]$$

<u>Agent i's perspective</u>

- I currently get $u_i(a_i) + s_i$
- If I swap places with agent j, I get $u_i(a_j) + s_j$
- I don't feel any happier, so I don't envy agent j

If we give a subsidy of \$3 to  and \$0 to  :

u  () + 3 = 10 = u  () + 0

u  () + 0 = 10 ≥ 7 = u  () + 3

Envy-free allocation with s bsidi

Not all allocations can be made envy-free!

- Allocation **a** = (a_1, …, a_n) , where ea
- Subsidy vector **s** = (s_1, …, s_n), w
- Outcome (**a**, **s**) is envy-free if

$u_i(a_i) + s_i \geq u_i(a_j) + s_j$, for eve air agents i, j ∈ [n]

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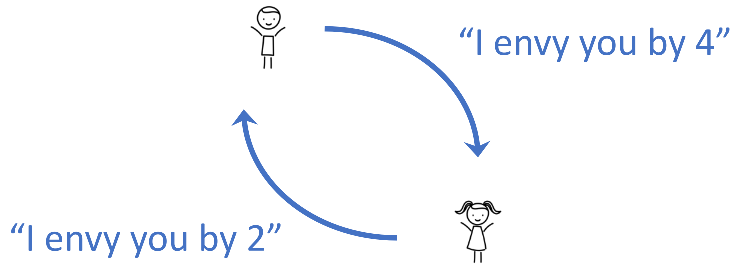

Envy-free allocation with s…bsidi…

- Allocation **a** = $(a_1, \ldots, a_n)$, where ea…
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- Outcome (**a**, **s**) is envy-free if

$u_i(a_i) + s_i \geq u_i(a_j) + s_j$, for eve… …air … agents $i, j \in [n]$

Not all allocations can be made envy-free!

		a_1	a_2
Agent 1	10	3	7
Agent 2	10	6	4

$3 + s_1 = u_1(a_1) + s_1 \geq u_1(a_2) + s_2 = 7 + s_2$
$4 + s_2 = u_2(a_2) + s_2 \geq u_2(a_1) + s_1 = 6 + s_1$

Since $3 + s_1 \geq 7 + s_2$ and $4 + s_2 \geq 6 + s_1$, we see that
$s_1 \geq (7-3) + s_2 \geq (7-3) + (6-4) + s_1 = 6 + s_1$
i.e. $s_1 \geq 6 + s_1 \Leftrightarrow 0 \geq 6$ (Impossible)

The 3 most relevant prior works

- [Jiarui **G**an, Warut **S**uksompong, Alexandros A **V**oudouris; 2019]
    - There is a polynomial time algorithm to check if there is an envy-free allocation
    - If such an envy-free allocation exists, output it

[GSV19] Jiarui Gan, Warut Suksompong, Alexandros A Voudouris. *Envy-freeness in house allocation problems*. Mathematical Social Sciences, 2019.

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  - Studied fair division of goods with subsidies + additive utilities
    - Complete allocation of items without m ≥ n restriction; agents can receive 0, or >1 good

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  - Definition: Envy-freeable (Informal: "Can find subsidy vector that works")
    - An allocation of goods is envy-freeable if there is a subsidy vector such that all agents are envy-free given their items' value(s) + subsidy

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    - An allocation of goods is envy-freeable if there is a subsidy vector such that all agents are envy-free given their items' value(s) + subsidy
  - There is a characterization of envy-freeable allocations
    - Implies that an envy-freeable allocation always exists for the house allocation problem
  - Given an envy-freeable allocation, there is a polynomial time algorithm to compute the unique corresponding subsidy vector that minimizes $\sum_i s_i$

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- [Siddharth **B**arman, Anand **K**rishna, Y. **N**arahari, Soumyarup **S**adhukhan; 2022]
  - If $(\mathbf{a}, \mathbf{s})$ is envy-free outcome, then so is $(\mathbf{a}_\sigma, \mathbf{s}_\sigma)$ for any permutation σ whenever $\mathbf{a}_\sigma$ is envy-freeable

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[BKNS22] Siddharth Barman, Anand Krishna, Y. Narahari, Soumyarup Sadhukhan. *Achieving Envy-Freeness with Limited Subsidies under Dichotomous Valuations*. International Joint Conference on Artificial Intelligence (IJCAI), 2022

Question

Given a house allocation problem instance, how do we find a minimum total subsidy allocation outcome?

(Remark: 0 total subsidy = Envy-free)

- Recall from prior works:
  - [GSV19] There is a polynomial time algorithm to check if an envy-free allocation exists, and output one if it exists
  - [HS19] Given an envy-freeable allocation (always exists), there is a poly time algorithm to compute the unique corresponding minimum total subsidy vector
  - [BKNS22] If ($\mathbf{a}$, $\mathbf{s}$) is envy-free outcome, then so is ($\mathbf{a}_\sigma$, $\mathbf{s}_\sigma$) for any permutation σ whenever $\mathbf{a}_\sigma$ is envy-freeable

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Minimum-subsidy envy-free outcome is NP-hard

- Reduction from Vertex Cover
  - n = $|V|^4 + |V|^3 + |E|$ agents
  - m = $|V|^4 + |V|^3 + |V|^2$ houses
  - Vertex cover size ≤ k ⇔ Total subsidy ≤ $\frac{k}{|V|}$

		Houses		
		Special ($\lvert V\rvert^4$)	Vertex v_{good} ($\lvert V\rvert$ for each v)	Vertex v_{bad} ($\lvert V\rvert^2$ for each v)
Agents	Special ($\lvert V\rvert^4$)	1	0	0
	Vertex w ($\lvert V\rvert^2$ for each $w \in V$)	0	$\begin{cases} 1+\lvert V\rvert^{-3} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$
	Edge $e = \{x, y\}$ (1 for each $e \in E$)	1	$\begin{cases} 1 & \text{if } v \in \{x, y\} \\ 0 & \text{otherwise} \end{cases}$	0

- Since any subset of n-1 vertices is a vertex cover, may assume that k < |V| – 1

Vertex cover size ≤ k ⇒ Total subsidy ≤ $\frac{k}{|V|}$

- Suppose C ⊆ V is a vertex cover with |C| ≤ k

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	Edge $e = \{x, y\}$ (1 for each $e \in E$)	1	$\begin{cases} 1 & \text{if } v \in \{x, y\} \\ 0 & \text{otherwise} \end{cases}$	0

Vertex cover size ≤ k $\Rightarrow$ Total subsidy ≤ $\dfrac{k}{|V|}$

- Suppose $C \subseteq V$ is a vertex cover with $|C| \le k$
- Proposed allocation
  - Assign each special agent to special house
  - Assign each vertex agent of type v to vertex house v_{bad}

		Houses		
		Special ($\lvert V\rvert^4$)	Vertex v_{good} ($\lvert V\rvert$ for each v)	Vertex v_{bad} ($\lvert V\rvert^2$ for each v)
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- Suppose $C \subseteq V$ is a vertex cover with $|C| \leq k$
- Proposed allocation
  - Assign each special agent to special house
  - Assign each vertex agent of type v to vertex house v_{bad}
  - For each edge agent corresponding to edge $\{x, y\}$, at least x or y must be in C
    - If $x \in C$, assign edge agent $\{x, y\}$ to x_{good}
    - If $y \in C$, assign edge agent $\{x, y\}$ to y_{good}
    - If both x and y are in C, assign arbitrarily

Always possible since there are |V| good houses for each vertex

Observation: In this allocation, only vertex agents v can possibly envy edge agents {v, ·}. No one else envies anyone else.

Vertex cover size ≤ k ⇒ Total subsidy ≤ $\frac{k}{|V|}$

		Houses		
		Special ($|V|^4$)	Vertex v_{good} ($|V|$ for each v)	Vertex v_{bad} ($|V|^2$ for each v)
Agents	Special ($|V|^4$)	1	0	0
Agents	Vertex w ($|V|^2$ for each $w \in V$)	0	$\begin{cases} 1+|V|^{-3} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$
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- Suppose C ⊆ V is a vertex cover with |C| ≤ k
- Proposed allocation
  - Assign each special agent to special house
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    - If x ∈ C, assign edge agent $\{x, y\}$ to x_{good}
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    - If both x and y are in C, assign arbitrarily

Always possible since there are |V| good houses for each vertex

- Proposed subsidy
  - If v ∈ C, give $|V|^{-3}$ to each vertex agent of type v
  - Give 0 to everyone else

$$\sum_i s_i = \frac{|V|^2 \cdot |C|}{|V|^3} = \frac{|C|}{|V|} \leq \frac{k}{|V|}$$

Observation: This subsidy of $|V|^{-3}$ does not create new envy since $1 > 0 + |V|^{-3}$

Vertex cover size ≤ k $\Leftarrow$ Total subsidy ≤ $\frac{\mathrm{k}}{|V|}$

- Suppose outcome (**a**, **s**) is envy-free outcome with $\sum_i s_i \leq \frac{\mathrm{k}}{|V|}$
- Define T = { v ∈ V : ∃ edge agent receiving house of type v_{good} in **a** }
- Claim 1: T is a vertex cover
- Claim 2: |T| ≤ k

To show

Vertex cover size ≤ k $\Leftarrow$ Total subsidy ≤ $\frac{k}{|V|}$

- Suppose outcome (**a**, **s**) is envy-free outcome with $\sum_i s_i \leq \frac{k}{|V|}$
- Define T = { v ∈ V : ∃ edge agent receiving house of type v_{good} in **a** }
- Claim 1: T is a vertex cover
  - n = $|V|^4 + |V|^3 + |E| > |V|^3 + |V|^2 = m - |V|^4$
  - Since n > m - $|V|^4$, by pigeonhole principle, some special house is allocated
  - If special agent **not** assigned special house, need to give subsidy of 1

- n = $|V|^4 + |V|^3 + |E|$ agents
- m = $|V|^4 + |V|^3 + |V|^2$ houses

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  - n = $|V|^4 + |V|^3 + |E| > |V|^3 + |V|^2 = m - |V|^4$
  - Since $n > m - |V|^4$, by pigeonhole principle, some special house is allocated
  - If special agent **not** assigned special house, need to give subsidy of 1
  - $k < |V| - 1 \Rightarrow \sum_i s_i \leq \frac{\mathrm{k}}{|V|} < \frac{|V|-1}{|V|} < 1 \Rightarrow$ *Any* agent's s_i subsidy is < 1
  - So, it must be the case that all special agents are assigned the special houses

Vertex cover size ≤ k $\Leftarrow$ Total subsidy ≤ $\frac{\mathrm{k}}{|V|}$

- Suppose outcome (**a**, **s**) is envy-free outcome with $\sum_i s_i \leq \frac{\mathrm{k}}{|V|}$
- Define T = { v ∈ V : ∃ edge agent receiving house of type $\mathrm{v}_{\mathrm{good}}$ in **a** }
- Claim 1: T is a vertex cover
    - All special agents are assigned all the special houses

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- Define T = { v ∈ V : ∃ edge agent receiving house of type v_{good} in **a** }
- Claim 1: T is a vertex cover
  - All special agents are assigned all the special houses
  - For edge agent $\{x, y\}$ to require < 1 subsidy, must assign x_{good} or y_{good}
  - This means that $T \cap \{x, y\} \neq \emptyset$ for any edge $\{x, y\} \in E$
  - That is, T is a vertex cover

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Vertex cover size ≤ k $\Leftarrow$ Total subsidy $\leq \frac{k}{|V|}$

- Suppose outcome (**a**, **s**) is envy-free outcome with $\sum_i s_i \leq \frac{k}{|V|}$
- Define T = { v ∈ V : ∃ edge agent receiving house of type v_{good} in **a** }
- Claim 1: T is a vertex cover
- Claim 2: |T| ≤ k
  - For any v ∈ T,
    - There is some edge agent receiving v_{good} (def^{n} of T)

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Vertex cover size ≤ k $\Leftarrow$ Total subsidy $\leq \frac{\mathrm{k}}{|V|}$

- Suppose outcome (**a**, **s**) is envy-free outcome with $\sum_i s_i \leq \frac{\mathrm{k}}{|V|}$
- Define T = { v ∈ V : ∃ edge agent receiving house of type v_{good} in **a** }
- Claim 1: T is a vertex cover
- Claim 2: |T| ≤ k
  - For any v ∈ T,
    - There is some edge agent receiving v_{good} (def^n of T)
    - Need to give vertex agent of type v either v_{good} or v_{bad}
    - If assigned v_{bad}, need to also give subsidy of $|V|^{-3}$

		Houses		
		Special ($\lvert V\rvert^4$)	Vertex v_{good} ($\lvert V\rvert$ for each v)	Vertex v_{bad} ($\lvert V\rvert^2$ for each v)
Agents	Special ($\lvert V\rvert^4$)	1	0	0
	Vertex w ($\lvert V\rvert^2$ for each $w \in V$)	0	$\begin{cases} 1+\lvert V\rvert^{-3} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$
	Edge $e = \{x, y\}$ (1 for each $e \in E$)	1	$\begin{cases} 1 & \text{if } v \in \{x, y\} \\ 0 & \text{otherwise} \end{cases}$	0

Recall: *Any* agent's subsidy is < 1

Vertex cover size ≤ k $\Leftarrow$ Total subsidy ≤ $\frac{k}{|V|}$

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- Claim 1: T is a vertex cover
- Claim 2: |T| ≤ k
  - For any v ∈ T,
    - There is some edge agent receiving v_{good} (def^n of T)
    - Need to give vertex agent of type v either v_{good} or v_{bad}
    - If assigned v_{bad}, need to also give subsidy of $|V|^{-3}$
    - There are $|V|^2$ vertex agents of type v but only |V| v_{good} houses (some are already taken)
  - So, total subsidy is *at least* $|T| \cdot (|V|^2 - |V|) \cdot |V|^{-3}$

		Houses		
		Special ($|V|^4$)	Vertex v_{good} ($|V|$ for each v)	Vertex v_{bad} ($|V|^2$ for each v)
Agents	Special ($|V|^4$)	1	0	0
	Vertex w ($|V|^2$ for each $w \in V$)	0	$\begin{cases} 1+|V|^{-3} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$
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Vertex cover size ≤ k $\Leftarrow$ Total subsidy ≤ $\frac{\mathrm{k}}{|V|}$

- Suppose outcome (**a**, **s**) is envy-free outcome with $\sum_i s_i \leq \frac{\mathrm{k}}{|V|}$
- Define T = { v ∈ V : ∃ edge agent receiving house of type $\mathrm{v}_{\mathrm{good}}$ in **a** }
- Claim 1: T is a vertex cover
- Claim 2: |T| ≤ k
  - Total subsidy is *at least* $|T| \cdot (|V|^2 - |V|) \cdot |V|^{-3}$
  - Suppose, for a contradiction, that $|T| \geq k + 1$. Then,

Since k < |V| - 1

$$\sum_i s_i \geq \frac{|T| \cdot (|V|^2 - |V|)}{|V|^3} \geq \frac{(k+1) \cdot (|V|^2 - |V|)}{|V|^3} = \frac{1}{|V|} \cdot \left(k + 1 - \frac{k+1}{|V|}\right) > \frac{k}{|V|}$$

  - Contradiction, so $|T| \leq k$

Minimum-subsidy envy-free outcome is NP-hard

- Reduction from Vertex Cover
  - Vertex cover size ≤ k ⇔ Total subsidy ≤ $\frac{\mathrm{k}}{|V|}$

		Houses		
		Special ($\lvert V\rvert^4$)	Vertex v_{good} ($\lvert V\rvert$ for each v)	Vertex v_{bad} ($\lvert V\rvert^2$ for each v)
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	Edge $e = \{x, y\}$ (1 for each $e \in E$)	1	$\begin{cases} 1 & \text{if } v \in \{x, y\} \\ 0 & \text{otherwise} \end{cases}$	0

  - Modifying $\hat{u}_i(h) = u_i(h) + c_i$, for some $c_i \geq 0$, does not affect envy-freeness
  - So, the NP-hardness argument holds even for normalized utilities where we have the same value of $\sum_h u_i(h)$ for all agents, after accounting for the c_i's

Two tractable cases

1) Identical valuations / utility functions

2) Similar number of agents and houses

Two tractable cases

1) **Identical valuations / utility functions**
    - u_i(any item) = u_j(same item) for all i, j ∈ [n]
    - Without loss of generality, by relabelling,
        - $u(h_1) \geq u(h_2) \geq \ldots \geq u(h_m)$
        - Agent i is assigned the i^{th} most valuable house within the subset of assigned houses

2) Similar number of agents and houses

Tractable case 1: Identical valuations

- **Observation 1**: Subsidy required is exactly the sum of value differences to the most valuable assigned house

	10	3	2
	10	3	2

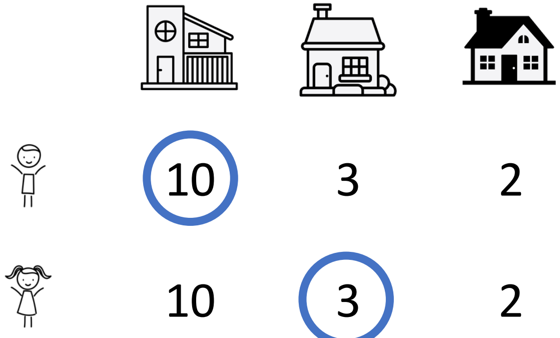
Envy-free is agent 2 given subsidy of 7

	10	3	2
	10	3	2

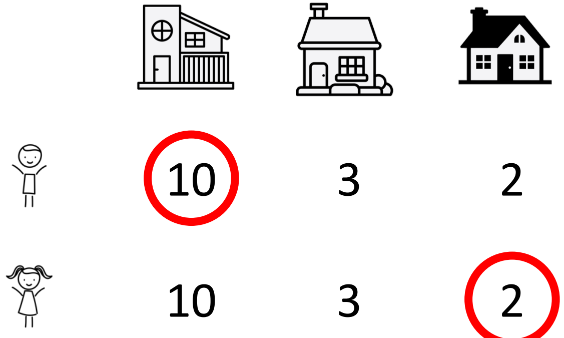
Envy-free if agent 2 given subsidy of 8

Tractable case 1: Identical valuations

- **Observation 1**: Subsidy required is exactly the sum of value differences to the most valuable assigned house
- **Observation 2**: For any fixed “most valuable assigned house”, we should always assign the contiguous n-1 houses right after it



Envy-free is agent 2 given subsidy of 7



Envy-free if agent 2 given subsidy of 8

Tractable case 1: Identical valuations

- **Observation 1**: Subsidy required is exactly the sum of value differences to the most valuable assigned house
- **Observation 2**: For any fixed “most valuable assigned house”, we should always assign the contiguous n-1 houses right after it
- Polynomial time algorithm to compute minimum subsidy allocation
    1. Compute prefix sums of values so we can compute required subsidy
    2. Check through all m-n “most valuable assigned house”
    3. Output the best option

Two tractable cases

1) Identical valuations / utility functions

2) Similar number of agents and houses

Two tractable cases

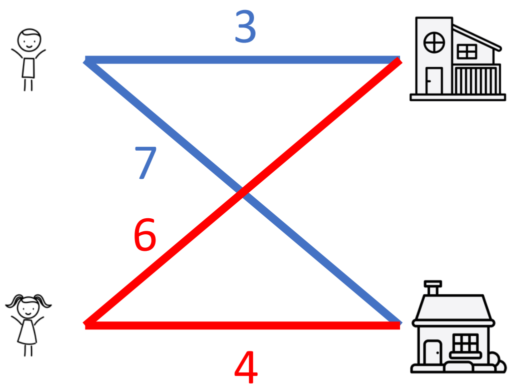
1) Identical valuations / utility functions

2) **Similar number of agents and houses**
   - m = n + c, for some constant c ≥ 0
   - Since $\binom{m}{n} = \binom{n+c}{n} = \binom{n+c}{c} \in O(n^c)$ is polynomial for constant c ≥ 0, suffice to show that the case of m = n can be solved in polynomial time

Tractable case 2: m = n

- Consider weighted complete bipartite graph G
  - Left partite: Agents
  - Right partite: Houses
  - Edge weights: $u_i(h_j)$, agent i's utility for house j
  - A perfect matching corresponds to an allocation

	3	7
	6	4



Tractable case 2: m = n

- Consider weighted complete bipartite graph G
- [HS19] Maximum weight perfect matching in G $\Leftrightarrow$ Envy-freeable allocation **a**
- Suppose **a** can be made envy-free with minimum subsidy vector **s**

Tractable case 2: m = n

- Consider weighted complete bipartite graph G
- [HS19] Maximum weight perfect matching in G ⇔ Envy-freeable allocation **a**
- Suppose **a** can be made envy-free with minimum subsidy vector **s**

Maybe some other allocation has a smaller subsidy vector?

Tractable case 2: m = n

- Consider weighted complete bipartite graph G
- [HS19] Maximum weight perfect matching in G ⇔ Envy-freeable allocation **a**
- Suppose **a** can be made envy-free with minimum subsidy vector **s**
- Since m = n, any envy-free allocation is a permutation of **a**
- [BKNS22] ($\mathbf{a}_\sigma$, $\mathbf{s}_\sigma$) is also envy-free for permutation σ if $\mathbf{a}_\sigma$ is envy-freeable
- Since **s** and $\mathbf{s}_\sigma$ are just permutations, the total subsidy is the same

Tractable case 2: $m = n$

- Consider weighted complete bipartite graph G
- [HS19] Maximum weight perfect matching in G ⇔ Envy-freeable allocation **a**
- Suppose **a** can be made envy-free with minimum subsidy vector **s**
- Since $m = n$, any envy-free allocation is a permutation of **a**
- [BKNS22] ($\mathbf{a}_\sigma$, $\mathbf{s}_\sigma$) is also envy-free for permutation σ if $\mathbf{a}_\sigma$ is envy-freeable
- Since **s** and $\mathbf{s}_\sigma$ are just permutations, the total subsidy is the same
- Polynomial time algorithm to compute minimum subsidy allocation
    1. Compute maximum weight perfect matching in G to get allocation **a**
    2. Compute corresponding minimum total subsidy vector **s** in polynomial time [HS19]
    3. Output (**a**, **s**)

Conclusion and future directions

- NP-hard in general to compute minimum subsidy envy-free allocation
- 2 tractable cases
  - All agents have identical utilities
  - Similar number of houses and agents ($m = n + c$, for constant $c \geq 0$)

Conclusion and future directions

- NP-hard in general to compute minimum subsidy envy-free allocation
- 2 tractable cases
  - All agents have identical utilities
  - Similar number of houses and agents ($m = n + c$, for constant $c \geq 0$)
- Conjecture: Polynomial time possible if identical preferences

	10	4	1
	9	3	3

Distinct utility functions but same preference ordering

Maybe "contiguous" observation also holds?

Conclusion and future directions

- NP-hard in general to compute minimum subsidy envy-free allocation
- 2 tractable cases
  - All agents have identical utilities
  - Similar number of houses and agents ($m = n + c$, for constant $c \geq 0$)
- Conjecture: Polynomial time possible if identical preferences
- Design approximation algorithms or prove hardness?
- Other notions of fairness? Pareto efficiency?
- Strategic behavior?
  - No deterministic mechanism can be strategy-proof (See Example 5.1 in paper)

Lying about own utility function helps

BACK UP SLIDES

Polynomial time algorithm for computing minimum subsidy vector

- Given allocation $\mathbf{a} = (a_1, \dots, a_n)$, compute envy graph $G_\mathbf{a}$
  - Vertices correspond to agents
  - Edges are directed and weighted
  - Weight of edge $i \rightarrow j$ is $u_i(a_j) - u_i(a_i)$, i.e. how much agent i envies agent j's allocation
  - Note that edge weights can be negative
- Define $\ell(i,j)$ as maximum weight of any path in $G_\mathbf{a}$ starting from i and ending at j
- Define $\ell(i) = \max_{j \in [n]} \ell(i,j)$
- [HS19, Theorem 2] $\mathbf{s} = (\ell(1), \dots, \ell(n))$ is the unique minimum total subsidy vector

Characterization of envy-freeable allocations

- [HS19, Theorem 1] The following are equivalent:
  - Allocation **a** = $(a_1, \dots , a_n)$ is envy-freeable
  - Allocation **a** maximizes utilitarian welfare across all reassignments

$$\sum_i u_i(a_i) \geq \sum_i u_i(a_{\sigma(i)}), \text{ for any permutation } \sigma$$

  - Envy graph $G_{\mathbf{a}}$ has no positive-weight cycles

For house allocation ($m = n$), the second condition corresponds to maximum weight perfect matching